Vandermonde determinant

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1 Definition

Definition 1.1. *Vandermonde Matrix*

A Vandermonde matrix, is a matrix with the terms of a geometric progression in each row, i.e., an $m \times n$ matrix

$$V^T = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{bmatrix}$$

or $V^T_{i,j} = a_i^{j-1}$.

We may also use the transpose form, which can be the coefficient matrix of some polynomial equations.

$$V = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_m \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \dots & \alpha_m^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \dots & \alpha_m^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \alpha_3^{n-1} & \dots & \alpha_m^{n-1} \end{bmatrix}$$

Theorem 1.2. *Determinant of square Vandermonde matrix*

$$det(V) = \prod_{1 \le i < j \le n} (a_i - a_j)$$

Proof. The following transformation doesn't change the determinant:

$$\text{for } (i = 1; i! = n; ++i)$$
$$\text{for } (j = n-1; j! = i-1; --j)$$
$$V \xrightarrow{R_{j+1} - a_i R_j} transformed(V)$$

The final matrix would be:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 & a_4 - a_1 & \dots & a_n - a_1 \\ 0 & 0 & (a_3 - a_1)(a_3 - a_2) & (a_4 - a_2)(a_4 - a_1) & \dots & (a_n - a_1)(a_n - a_2) \\ 0 & 0 & 0 & (a_4 - a_1)(a_4 - a_2)(a_4 - a_3) & \dots & (a_n - a_1)(a_n - a_2)(a_n - a_3) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (a_n - a_1)(a_n - a_2)\dots(a_n - a_{n-1}) \end{bmatrix}$$

Hence

$$det(V) = \prod_{1 \le i < j \le n} (a_i - a_j)$$

□

Vandermonde matrix is closely related to the polynomial and differentiation when used with **L'Hospital's rule**.

Example 1.2.1. *Putnam 1986 A6*

Let $a_1, a_2, \dots, a_n$ be real numbers, and let $b_1, b_2, \dots, b_n$ be distinct positive integers. Suppose there is a polynomial $f(x)$ satisfying the identity

$$(1 - x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \dots, b_n$ and n (but independent of $a_1, a_2, \dots, a_n$).

Answer:

$$f(x) = \frac{\prod_{i=1}^{n} b_i}{n!}$$

Proof. Applying **L'Hospital's rule** to $f(1)$, we have:

$$(1 - x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}$$

$$f(1) = \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1 + \sum_{i=1}^{n} a_i x^{b_i}}{(1 - x)^n}$$

$$f^{(i)}(x) = \begin{cases} -1 & (i = 0) \\ 0 & (0 < i < n) \\ x & (i = n) \end{cases}$$

We can get the matrix for $a_1, a_2, \dots, a_n$:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & \dots & 1 & -1 \\ b_1 & b_2 & b_3 & \dots & b_n & 0 \\ b_1(b_1 - 1) & b_2(b_2 - 1) & b_3(b_3 - 1) & \dots & b_n(b_n - 1) & 0 \\ b_1(b_1 - 1)(b_1 - 2) & b_2(b_2 - 1)(b_2 - 2) & b_3(b_3 - 1)(b_3 - 2) & \dots & b_n(b_n - 1)(b_n - 2) & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 \dots (b_1 - n + 1) & b_2 \dots (b_2 - n + 1) & b_3 \dots (b_3 - n + 1) & \dots & b_n \dots (b_n - n + 1) & x \end{array} \right)$$

Via elementary row operation from R_2 to R_{n+1}, the above matrix is equivalent to:

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & \dots & 1 & -1 \\ b_1 & b_2 & b_3 & \dots & b_n & 0 \\ b_1^2 & b_2^2 & b_3^2 & \dots & b_n^2 & 0 \\ b_1^3 & b_2^3 & b_3^3 & \dots & b_n^3 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1^n & b_2^n & b_3^n & \dots & b_n^n & x \end{array}\right)$$

Following the procedures we get the determinant of Vandermonde matrix, we have:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & b_2-b_1 & b_3-b_1 & b_4-b_1 & \dots & b_n-b_1 \\ 0 & 0 & (b_3-b_1)(b_3-b_2) & (b_4-b_2)(b_4-b_1) & \dots & \\ 0 & 0 & 0 & (b_4-b_1)(b_4-b_2)(b_4-b_3) & \dots & (b_n-b_1)(b_n-b_2)(b_n-b_3) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (b_n-b_1)(b_n-b_2)\dots(b_n-b_{n-1}) \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ b_1 \\ -b_1b_2 \\ b_1b_2b_3 \\ \vdots \\ (-1)^n b_1b_2\dots b_n \\ x-(-1)^n b_1b_2\dots b_n \end{pmatrix}$$

So $x-(-1)^n b_1b_2\dots b_n = 0$ Since $(1-x)^{(n)} = (-1)^n n!$, we have the answer. □

Notice the important application of **L'Hospital's Rule**, it is extremely useful to determine a polynomial.

Example 1.2.2. *MAS941 Problem Set 7 Question 5*

Prove that, suppose $n \neq 2^k$, if we know the pairwise sum of n numbers, we can deduce the n summands.

Proof. If there's only one solution, we can get the n summands via the following algorithm:
Algorithm
Denote the n summands are $a_1, a_2, \dots, a_n$, the $\frac{n(n-1)}{2}$ known numbers are $x_1, x_2, \dots, x_{\frac{n(n-1)}{2}}$.

1. Sort the $\frac{n(n-1)}{2}$ number. (Notice that $x_1 = a_1 + a_2, x_2 = a_1 + a_3$, we have:)
2. Set i=3,$x_i = a_1 + a3$
3. Set the left minimum as $a_1 + a_4$, we can deduce all $the a_j + a_4$
4. Set the left minimum as $a_1 + a_5$, …

5. ...

6. If we get all the numbers, the algorithms halts. Otherwise, $i++$, Goto Step 2.

Otherwise, suppose there are two solution set $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$. We have: For $\forall t$,

$$\left(\sum_{i=1}^{n} a_i\right)^2 - \left(\sum_{i=1}^{n} b_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2 - b_i^2\right)$$

$\square$